PRIMARY CYCLES ON THE CIRCLE

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ABSTRACT. We consider cycles, i.e., periodic orbits, of continuous degree one maps of the circle. One cycle forces another if every such map that has a cycle which looks like the first also has a cycle which looks like the second. We characterize primary cycles, i.e., those which force no other cycle of the same period.

INTRODUCTION

This paper concerns cycles, i.e., periodic orbits, of continuous degree one maps of the circle. As is customary in this subject, we deal with lifts of maps and cycles to the reals rather than with maps and cycles themselves.

We say that $P \subseteq \mathbf{R}$ is a *cycle* if it has the properties of a lift of a cycle of a continuous degree one map of the circle, and that P *forces* Q if every map that has a cycle which looks like P also has one which looks like Q. (See §2 for precise definitions.) "Looks like" is really an equivalence relation on cycles, and the proper formal setting for forcing is equivalence classes of cycles, rather than cycles themselves. However, most of the proofs use actual cycles, so we sacrifice the precision of equivalence classes in order to avoid an additional layer of notation.

We show that "forces" is a partial order on (equivalence classes of) cycles and that each cycle forces only finitely many inequivalent cycles of a given period. Thus for each $n > 0$, there are cycles of period n which force no inequivalent cycle of period n. We call such cycles *primary*. The main results of this paper are Theorem 4.1, which characterizes primary cycles, and Theorem 3.15, which characterizes cycles having a trivial rotation interval.

Forcing for maps of the interval has been studied by a number of authors, most systematically by S. Baldwin [Ba]. L. Alsedà, J. Llibre and R. Serra [ALS], L. Block and W. A. Coppel [BC], Coppel [C], and C.-W. Ho [H] all deal with the problem of characterizing primary interval cycles. We summarize their results in §1. Forcing, with emphasis on primary cycles, for maps of the interval and of the space $Y = \{z \in \mathbf{C}: 0 \le z^3 \le 1\}$ has been studied by Alsedà, Llibre and M. Misiurewicz [ALM]. Special cases of forcing for maps of the circle have been considered by Alsedà, Llibre, Misiurewicz and C. Simó [ALMS], A. Chenciner,

J.-M. Gambaudo and Ch. Tresser [CGT], Llibre [L], R. MacKay and Tresser [MT], Misiurewicz [M2], and S. Newhouse, J. Palis and F. Takens [NPT].

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1. BACKGROUND FROM THE INTERVAL

Definitions. (1) A(n interval) *cycle of period* n is a cyclic permutation $\alpha\colon X \to X = \{x_1 < \cdots < x_n\} \subseteq \mathbf{R}$ of an n-element set of real numbers. We often suppress α and refer to X as the cycle. The period of X is denoted by $|X|$.

(2) Cycles $\alpha\colon X \to X$ and $\beta\colon Y \to Y$ are *equivalent* if $|X| = |Y|$, and $\alpha(x_i) = x_j$ if and only if $\beta(y_i) = (y_j)$.

(3) An *interval map* is a continuous map $f\colon I \to I$ from a compact interval to itself.

(4) An *f-cycle* is a set $X \subseteq I$ such that $f|_X\colon X \to X$ is a cycle.

(5) X *forces* Y if every interval map which has a cycle equivalent to X also has one equivalent to Y.

Baldwin [Ba, Theorem 1.4] proved that "forces" is a partial order on the set of equivalence classes of cycles. See also [ALM].

Definition. A cycle X is *primary* if every cycle of period $|X|$ which is forced by X is equivalent to X.

Since there are only finitely many equivalence classes of a given period, it follows that every cycle forces a primary cycle of the same period.

Theorem 1.1 below characterizes primary cycles.

Definitions. (1) A cycle $\alpha\colon X \to X$ of period n has a *block structure* over a cycle $\beta\colon Y \to Y = \{y_1 < \cdots < y_m\}$ of period m if n is a multiple of m, say $n = sm$, and if X is written $X = X_1 \cup \cdots \cup X_m$, where $X_i = \{x_{(i-1)s+1} < \cdots < x_{(i-1)s+s}\}$, then $\alpha(X_i) = X_j$ if and only if $\beta(y_i) = y_j$.

(2) X is a *Z-extension* of Y if it has a block structure over Y as in (1), such that for some k, $1 \le k \le m$, α is monotone on X_i for all $i \ne k$, and $\alpha^m|_{X_k}\colon X_k \to X_k$ is equivalent to Z.

(3) A *2-extension* is one in which Z is a two-point cycle. (There is just one equivalence class of two-point cycles.)

(4) A *Štefan cycle* is a cycle $\alpha\colon X \to X$ of odd period $n \ge 3$, such that X can be written $\{\bar{x}_n < \bar{x}_{n-2} < \cdots < \bar{x}_3 < \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_{n-1}\}$, or with the inequalities reversed, and $\alpha(\bar{x}_1) = \bar{x}_2, \ldots, \alpha(\bar{x}_{n-1}) = \bar{x}_n$, and $\alpha(\bar{x}_n) = \bar{x}_1$. (For each odd $n \ge 3$, there are exactly two equivalence classes of Štefan cycles.)

(5) A *Štefan extension* is one in which Z is a Štefan cycle.

Theorem 1.1 (cf. [ALM, Remark 9.2]). *The following statements about a cycle X are equivalent.*

(1) *X is primary.*

(2) *X can be obtained from the one-point cycle by a finite number (possibly zero) of 2-extensions and then at most one Štefan extension.*

(3) *There is an interval map whose only cycle of period $|X|$ is X.*

Proof. (1) ⇔ (2) is implicit in [Š] for $|X|$ odd and in [Bl] for $|X|$ a power of two. For $|X|$ even and not a power of two, (1) ⇔ (2) was conjectured by Baldwin [Ba, Conjecture 6.3], and (1) ⇒ (2) was proved by Block and Coppel [BC, Theorem 1].

Clearly (3) ⇒ (1). We show that (2) ⇒ (3). Let $\alpha\colon X \to X = \{x_1 < \cdots < x_n\}$. If $n = 1$, let f be the constant map $f(x) = x_1$, defined on any nondegenerate compact interval containing x_1. If $n \geq 2$ is a power of two, let $f\colon [x_1, x_n] \to [x_1, x_n]$ agree with α on X and be monotone quadratic on $[x_1, x_2]$ and linear on $[x_i, x_{i+1}]$ for $i \neq 1$. If n is not a power of two, let $f\colon [x_1, x_n] \to [x_1, x_n]$ agree with α on X and be linear on each $[x_i, x_{i+1}]$. In each case, it is easy to check, using the Markov graph, that X is the only f-cycle of period n. □

There is no standard terminology in this area. Cycles satisfying (1) are called *minimal* in [Ba] and *primary* in this paper, those satisfying (2) are called *strongly simple* in [BC], and those satisfying (3) are called *primary* in [ALM]. Cycles are often called (periodic) *orbits*.

2. CYCLES AND FORCING ON THE CIRCLE

Let f be a continuous degree one map of the circle, and let C be an f-cycle of period n. Let $\pi\colon \mathbf{R} \to \mathbf{R}/\mathbf{Z}$ be the canonical projection onto the circle and let F be a lift of f. Then, writing $\pi^{-1}(C) = \{\cdots < x_0 < x_1 < \cdots\}$, we have

(a) $x_{i+n} = x_i + 1$.

(b) For all $x \in \mathbf{R}$, $F(x + 1) = F(x) + 1$.

(c) For all $x, y \in \pi^{-1}(C)$, there are integers $m > 0$ and s such that $F^m(x) = y + s$.

(d) There is an integer r such that for all $x \in \pi^{-1}(C)$, $F^n(x) = x + r$.

(d) follows from (a), (b), and (c). See Lemma 2.1 below.

We abstract this situation, keeping only $F|_{\pi^{-1}(C)}$.

Definition. A (degree one) *cycle of period* n is a map $\varphi\colon P \to P = \{\cdots < x_0 < x_1 < \cdots\} \subseteq \mathbf{R}$ such that

(a) $x_{i+n} = x_i + 1$.

(b) For all $x \in P$, $\varphi(x + 1) = \varphi(x) + 1$.

(c) For all $x, y \in P$, there are integers $m > 0$ and s such that $\varphi^m(x) = y + s$.

We often suppress φ and refer to P as the cycle. The period of P is denoted by $|P|$.

Lemma 2.1. *Let $\varphi\colon P \to P$ be a cycle of period n. There is an integer r such that for all $x \in P$, $\varphi^n(x) = x + r$.*

Proof. Let $\widetilde{\varphi}(x) = \varphi(x) - [\varphi(x)]$, i.e., the fractional part of $\varphi(x)$. Then $\widetilde{\varphi}$ is a cyclic permutation of the n-element set $P \cap [0, 1)$. Therefore $\widetilde{\varphi}^n(x) = x$ for all $x \in P \cap [0, 1)$, and so for each $x \in P$, there is an integer r such that $\varphi^n(x) = x + r$. By (b) and (c), r does not depend on x. $\square$

Definitions. (1) The number $\rho(P) = r/n$ of Lemma 2.1 is called the *rotation number* of P. (Writing $\rho(P) = r/n$ does not imply that $\mathrm{GCD}(r, n) = 1$.)

(2) Cycles $\varphi: P \to P$ and $\psi: Q \to Q$ are *equivalent* if $|P| = |Q|$ and we can write $P = \{\cdots < x_0 < x_1 < \cdots\}$ and $Q = \{\cdots < y_0 < y_1 < \cdots\}$ in such a way that $\varphi(x_i) = x_j$ if and only if $\psi(y_i) = y_j$.

(3) Let $\mathscr{C}$ denote the class of continuous maps $F: \mathbf{R} \to \mathbf{R}$ which are lifts of continuous degree one maps of the circle; i.e., $F \in \mathscr{C}$ if and only if F is continuous and $F(x + 1) = F(x) + 1$ for all $x \in \mathbf{R}$. Thus $\mathscr{C}$ is the class of continuous old maps. ("Old" is an anagram of an acronym: "lift of degree one" [M3]. We pass up the chance to call the maps in $\mathscr{C}$ "cold.")

(4) Let $F \in \mathscr{C}$. An *F-cycle* is a set $P \subseteq \mathbf{R}$ such that $F|_P: P \to P$ is a cycle.

(5) P *forces* Q if every map in $\mathscr{C}$ which has a cycle equivalent to P also has a cycle equivalent to Q.

Note that φ and $\varphi + 1$ are not equivalent, although they represent the same cycle on the circle. Making φ and $\varphi + 1$ equivalent would negate the benefit of dealing with lifted cycles rather than cycles on the circle itself and would not allow meaningful comparison of rotation numbers of different cycles.

The main result of this section is Theorem 2.6, which states that "forces" is a partial order on equivalence classes of cycles.

Definition. Let $F \in \mathscr{C}$ and let P be an F-cycle. F is *P-monotone* if it is (not necessarily strictly) monotone between adjacent members of P.

Lemma 2.2. *Let F be P-monotone. Then P forces Q if and only if F has a cycle equivalent to Q.*

Proof. Whether P forces Q is determined by the oriented Markov graph of P and the equivalence classes of P and Q. (Cf. [Ba, Theorem 3.3], [Be, Lemma 1.8], or [ALM, Lemma 1.18], the corresponding result for interval cycles.) $\square$

Lemma 2.3. *Let $F \in \mathscr{C}$. If F is piecewise polynomial and each piece is of degree at least two, then F has only finitely many cycles of each period.*

Proof. Let $n > 0$ be given. Since $\{F^n(x) - x : x \in [0, 1]\}$ is compact, there are only finitely many integers r such that $F^n(x) = x + r$ has a solution in $[0, 1]$. Fix such an r. Since $F^n - id$ is piecewise polynomial, and none of its pieces is constant, there are only finitely many $x \in [0, 1]$ which satisfy $F^n(x) = x + r$. $\square$

Lemma 2.4. *For every cycle P, there is a P-monotone map F which has only finitely many cycles of each period.*

Proof. Let $\varphi: P \to P$. Let F agree with φ on P, be monotone quadratic on each $[x_i, x_{i+1}]$, and satisfy $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$. The result now follows from Lemma 2.3. $\square$

Definition. Let $F \in \mathscr{C}$, let $n > 0$ be an integer, and let $Q = \{ \cdots < y_0 < y_1 < \cdots \}$ be a set which satisfies $y_{i+n} = y_i + 1$ for each i. The *F-variation* of Q is $\operatorname{var}_F(Q) = \sum_{1 \le i \le n} |F(y_i) - F(y_{i-1})|$. (This number is well defined; i.e., it does not change if Q is renumbered.)

Lemma 2.5. *Let $F \in \mathscr{C}$ and let P be an F-cycle. If P forces Q but P and Q are not equivalent, then F has a cycle Q', equivalent to Q, such that $\operatorname{var}_F(Q') < \operatorname{var}_F(P)$.*

Proof. Let $P = \{ \cdots < x_0 < x_1 < \cdots \}$. Then $F|_P = \varphi$ is not increasing. For otherwise, if $\varphi^n(x) = x + r$ for all $x \in P$, then φ is equivalent to $\psi: \mathbf{Z}/n \to \mathbf{Z}/n = \{i/n: i \in \mathbf{Z}\}$, defined by $\psi(y) = y + r/n$, and hence by Lemma 2.2, P forces only cycles equivalent to itself.

Let $G \in \mathscr{C}$ be P-monotone. Since P forces Q, G has a cycle $R = \{ \cdots < z_0 < z_1 < \cdots \}$ equivalent to Q. As in the proofs of [Ba, Theorem 3.3], [Be, Lemma 1.8], and [ALM, Lemma 1.18], F has a cycle $Q' = \{ \cdots < y_0 < y_1 < \cdots \}$, equivalent to Q, labeled in such a way that for all i, j, k, $x_i < y_j < x_{i+1}$ if and only if $x_i < z_j < x_{i+1}$, and $F(y_j) = y_k$ if and only if $G(z_j) = z_k$. It follows that if $s \ge 0$ and $x_i < y_j < y_{j+1} < \cdots < y_{j+s} < x_{i+1}$, then either $F(x_i) < F(y_j) < \cdots < F(y_{j+s}) < F(x_{i+1})$ or $F(x_i) > F(y_j) > \cdots > F(y_{j+s}) > F(x_{i+1})$, because either $G(x_i) < G(z_j) < \cdots < G(z_{j+s}) < G(x_{i+1})$ or $G(x_i) > G(z_j) > \cdots > G(z_{j+s}) > G(x_{i+1})$. Thus $\operatorname{var}_F(P) = \operatorname{var}_F(P \cup Q')$. On the other hand, since $F|_P$ is not monotone, $\operatorname{var}_F(Q') < \operatorname{var}_F(P \cup Q')$. $\square$

Theorem 2.6. *"Forces" is a partial order on equivalence classes of cycles.*

Proof. Since "forces" is reflexive and transitive on equivalence classes, it suffices to show that it is antisymmetric; i.e., if P forces Q and Q forces P, then P is equivalent to Q.

Suppose that there are cycles P and Q such that P forces Q and Q forces P, but P is not equivalent to Q. By Lemma 2.4, there is a map $F \in \mathscr{C}$ for which P is an F-cycle and which has only finitely many cycles of each period. Applying Lemma 2.5 to F inductively, alternating the statements "P forces Q" and "Q forces P," we find that F has cycles $P = P_0, P_1, P_2, \ldots$, all equivalent to P, and cycles $Q_1, Q_2, \ldots$, all equivalent to Q, such that for each $i \ge 0$, $\operatorname{var}_F(P_i) > \operatorname{var}_F(Q_{i+1}) > \operatorname{var}_F(P_{i+1})$. This violates the choice of F. $\square$

3. Twist and primary cycles

Definition. A cycle $\varphi: P \to P$ is a *twist cycle* if φ is increasing.

The term "twist" is justified by the following fact, noted in the proof of Lemma 2.5. A twist cycle with period n and rotation number r/n is equivalent

to the cycle $\psi: \mathbf{Z}/n \to \mathbf{Z}/n$, defined by $\psi(x) = x + r/n$. If P is a twist cycle of period n and rotation number r/n, then $GCD(r,n) = 1$. In particular, there is only one equivalence class of twist cycles with a given rotation number.

Theorem 3.1. *Every cycle forces a twist cycle with the same rotation number.*

Proof. Let P be a cycle and let F be P-monotone. By [ALMS, Theorem A] (or [M2, Theorem B] or [CGT, Theorem 2]), F has a twist cycle with the same rotation number as P. □

Theorem 3.2 shows that twist cycles may be thought of as "prime."

Theorem 3.2. *The following statements about a cycle P are equivalent.*

(1) *P is a twist cycle.*
(2) *Every cycle with rotation number $\rho(P)$ which is forced by P is equivalent to P.*
(3) *Every cycle which is forced by P is equivalent to P.*
(4) *There is a map in $\mathscr{C}$ whose only cycle is P.*

Proof. $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ are obvious, and $(2) \Rightarrow (1)$ follows from Theorem 3.1. To prove $(1) \Rightarrow (4)$, let $\varphi: P \to P = \{\cdots < x_0 < x_1 < \cdots\}$ have rotation number r/n (in lowest terms). Then, as noted after the definition of twist cycle, $\varphi(x_i) = x_{i+r}$ for all i. Let F agree with φ on P and be monotone quadratic on $[x_i, x_{i+1}]$ if $i \equiv 0 \pmod{n}$ and linear on $[x_i, x_{i+1}]$ if $i \not\equiv 0 \pmod{n}$. □

Definition. A cycle P is *primary* if every cycle of period $|P|$ which is forced by P is equivalent to P.

Theorem 3.3. *P is primary if every cycle of period $|P|$ and rotation number $\rho(P)$ which is forced by P is equivalent to P.*

Before proving Theorem 3.3, we derive a corollary.

Corollary 3.4. *Every cycle forces a primary cycle with the same period and rotation number.*

Proof. Let P be a cycle and let $\mathscr{P}$ denote the set of cycles forced by P and having the same period and rotation number as P. Then $\mathscr{P} \neq \varnothing$ $(P \in \mathscr{P})$ and by Lemma 2.4, $\mathscr{P}$ is finite. Hence by Theorem 2.6, there is a cycle in $\mathscr{P}$ which is equivalent to any cycle in $\mathscr{P}$ that it forces. This cycle must be primary. □

We will prove Theorem 3.3 by examining rotation intervals [NPT, Chapter III, §3].

Definitions. Let $F \in \mathscr{C}$ and let $x \in \mathbf{R}$. The number

$$\rho(x) = \limsup_{n \to \infty} (F^n(x) - x)/n$$

is called the *rotation number* of x. The closed interval $[\rho_-(F), \rho_+(F)]$, where $\rho_-(F) = \inf\{\rho(x): x \in \mathbf{R}\}$ and $\rho_+(F) = \sup\{\rho(x): x \in \mathbf{R}\}$ is called the

rotation interval of F. When $\rho_-(F) = \rho_+(F)$, we denote this number by $\rho(F)$.

If a map in $\mathscr{C}$ is nondecreasing, then its rotation interval is trivial. The converse is false. (See the example near the end of this section.)

Lemma 3.5 [NPT, Chapter III, Corollary 3.2]. *If $\rho_-(F) \leq r/n \leq \rho_+(F)$, then F has a cycle with rotation number r/n.*

Definition. For a cycle P, let $\rho_-(P) = \rho_-(F)$ and $\rho_+(P) = \rho_+(F)$, where F is any P-monotone map. (These numbers are well defined.) The *rotation interval* of P is $[\rho_-(P), \rho_+(P)]$.

We show in Theorem 3.14 that primary cycles have trivial rotation intervals. Theorem 3.15 characterizes cycles which have trivial rotation intervals.

Lemma 3.6. $\rho_-(P) = \inf\{\rho(Q): P \text{ forces } Q\}$ *and* $\rho_+(P) = \sup\{\rho(Q): P \text{ forces } Q\}$.

Proof. Apply Lemma 3.5. $\square$

Another way to obtain the rotation interval of F is to introduce (as in [ALMS]) the maps F_- and F_+.

Definition. $F_-(x) = \inf_{y \geq x} F(y)$ and $F_+(x) = \sup_{y \leq x} F(y)$.

Note that $F_- \leq F \leq F_+$, $F_-, F_+ \in \mathscr{C}$, and F_- and F_+ are nondecreasing and hence have trivial rotation intervals. For each $n > 0$, $(F_-)^n = (F^n)_-$ and $(F_+)^n = (F^n)_+$. Hence the symbols F_-^n and F_+^n are unambiguous.

Lemma 3.7. $\rho_-(F) = \rho(F_-)$ *and* $\rho_+(F) = \rho(F_+)$.

Proof. A routine exercise. $\square$

Definition. The *nonconstant part* of a map is the complement of the union of the open intervals on which the map is constant.

Lemma 3.8. *Let F be P-monotone and satisfy either $\rho(P) = \rho_-(P)$ or $\rho(P) = \rho_+(P)$. Then F has a twist cycle Q such that*

    (1) $\rho(P) = \rho(Q)$.

    (2) *For $x \in P$ and $y \in Q$, $x < y$ if and only if $F(x) < F(y)$.*

Proof. Let $\rho(P) = \rho_+(P) = r/n$, where $n = |P|$. By Lemma 3.5 (applied to F_+), there is a point $z \in \mathbf{R}$ such that $F_+^n(z) = z + r$. Since $F_+^n \in \mathscr{C}$, z may be chosen to lie on the nonconstant part of F_+^n. Then $z, F_+(z), \ldots, F_+^{n-1}(z)$ all lie on the nonconstant part of F_+. Therefore $F = F_+$ at these points. Thus there is an F-cycle Q which contains z. Since $F = F_+$ on Q and F_+ is nondecreasing, Q is a twist cycle. Clearly $\rho(Q) = r/n$.

Now let $x \in P$, $y \in Q$, with $x \neq y$. Then $F(x) \neq F(y)$. If $x < y$, then (since $F \leq F_+$, F_+ is nondecreasing, and $F_+ = F$ on Q) we have $F(x) \leq F_+(x) \leq F_+(y) = F(y)$. Conversely, suppose $x > y$ and $F(x) < F(y)$. Since $F^n(x) = x + r$ and $F^n(y) = y + r$, there is an integer k, $0 < k < n$, such that

$F^k(x) < F^k(y)$ and $F^{k+1}(x) > F^{k+1}(y)$. But $F^k(x) \in P$ and $F^k(y) \in Q$, so this can't happen.

If $\rho(P) = \rho_-(P) = r/n$, the same argument works with F_- in place of F_+ and the inequalities reversed. □

Lemma 3.9. *Let F be a P-monotone map which has a twist cycle Q such that for $x \in P$ and $y \in Q$, $x < y$ if and only if $F(x) < F(y)$. Then $\rho_-(P) = \rho_+(P)$.*

Proof. If $P = Q$, there is nothing to prove. So we assume that $P \neq Q$ and hence $P \cap Q = \varnothing$. It suffices to show

(∗)   For $z \in \mathbf{R}$ and $y \in Q$, $y < z$ if and only if $F(y) < F(z)$,

for then $\rho_-(F) = \rho_+(F) = \rho(Q)$.

Let $y < y'$ be consecutive points of Q. Because F is P-monotone and $P \cap Q = \varnothing$, $F[y, y']$ is the convex hull of the finite set $\{F(y), F(y')\} \cup \{F(x): x \in P \cap [y, y']\}$. But $F(y) < F(x) < F(y')$ for all $x \in P \cap [y, y']$. Hence $F[y, y'] = [F(y), F(y')]$, and (∗) follows. □

Lemmas 3.8 and 3.9 together yield a characterization of cycles with trivial rotation intervals. A more tractable characterization is given by Theorem 3.15 at the end of this section.

Lemma 3.10. *For any cycle P, either $\rho_-(P) < \rho(P) < \rho_+(P)$ or $\rho_-(P) = \rho(P) = \rho_+(P)$.*

Proof. Apply Lemmas 3.8 and 3.9. □

Lemma 3.11 (cf. [M1, Proposition 4; M2, Proposition 5.1]). *If $\rho_-(F) < r/n < \rho_+(F)$, then F has a cycle with period n and rotation number r/n which lies on the nonconstant part of F.*

Proof. Write $r/n = s/m$ where $\mathrm{GCD}(s, m) = 1$, and set $G = F^m - s$. Then $\rho_-(G) < 0 < \rho_+(G)$, and hence by Lemma 3.5, G has a cycle with a negative rotation number and one with a positive rotation number. Therefore, by [ALMS, Theorem A], G has a twist cycle P with a positive rotation number and a twist cycle Q with a negative rotation number. Choose $x \in P$ and $y \in Q$ so that $x < y$ and $(x, y) \cap (P \cup Q) = \varnothing$. Then $G(y) < x < y < G(x)$. Let $I = [G(y), x]$ and $J = [y, G(x)]$. Then each of I and J G-covers both I and J in an orientation-preserving manner. (We say that $[a, b]$ *G-covers $[c, d]$ in an orientation-preserving manner* if $G(a) \leq c$ and $G(b) \geq d$.)

Let $t = n/m$, and let $K_0 = I$, $K_1 = K_2 = \cdots = K_{t-1} = J$, $K_t = I$. (If $t = 1$, let $K_0 = K_1 = I$.) Let $L_t = K_t$, and inductively construct subintervals $L_i \subseteq K_i$, $t - 1 \geq i \geq 0$, such that L_i G-covers L_{i+1} in an orientation-preserving manner and $G(L_i) = L_{i+1}$. Then $z = \sup\{w \in L_0: G^t(w) = w\}$ has G-period t and lies on the nonconstant part of G^t, and hence $z, G(z), \ldots, G^{t-1}(z)$ all lie on the nonconstant part of G.

The F-cycle which contains z has rotation number r/n, and by [M1, Lemma 5] period $tm = n$. Since $z, G(z), \ldots, G^{t-1}(z)$ all lie on the nonconstant part

of G, they lie on the nonconstant part of F^m, and hence $z, F(z), \ldots, F^{n-1}(z)$ all lie on the nonconstant part of F. $\square$

Lemma 3.12. *Let* $\rho_-(F) \le a \le b \le \rho_+(F)$. *Then there exists a map* $G \in \mathscr{C}$ *which agrees with* F *on the nonconstant part of* G, *such that* $\rho_-(G) = a$ *and* $\rho_+(G) = b$.

Proof. For $t \ge 0$, set $F_t(x) = \min\{F(x), F_-(x) + t\}$. Note that $F_t \in \mathscr{C}$. Since F_- agrees with F on its nonconstant part, so does each F_t. Since the mappings $t \mapsto F_t$ and [NPT, Chapter III, Proposition 3.3] $G \mapsto \rho_-(G)$ and $G \mapsto \rho_+(G)$ are continuous (with spaces of maps having the C^0 topology), the mappings $t \mapsto \rho_-(F_t)$ and $t \mapsto \rho_+(F_t)$ are continuous. But $F_0 = F_-$, and for t large enough, $F_t = F$. Thus $\rho_+(F_0) \le a$, and for t large enough, $\rho_+(F_t) \ge b$. Therefore there exists $t_0 \ge 0$ such that $\rho_+(F_{t_0}) = b$. But $(F_t)_- = F_-$ for all $t \ge 0$, and so by Lemma 3.7, $\rho_-(F_{t_0}) = \rho_-(F)$.

Now make a similar construction, starting with F_{t_0} instead of F, and with the subscripts $+$ and $-$ interchanged. We obtain a map G which agrees with F_{t_0}, and hence with F, on its nonconstant part, such that $\rho_-(G) = a$ and $\rho_+(G) = \rho_+(F_{t_0})$, i.e., $\rho_+(G) = b$. $\square$

Lemma 3.13. *Let* P *be a cycle such that every cycle with period* $|P|$ *and rotation number* $\rho(P)$ *which is forced by* P *is equivalent to* P. *Then* $\rho_-(P) = \rho_+(P)$.

Proof. Let P have period n and rotation number r/n, and let F be P-monotone.

Suppose $\rho_-(P) \ne \rho_+(P)$. Then by Lemma 3.10, $\rho_-(P) < r/n < \rho_+(P)$. Choose a and b so that $\rho_-(P) < a < r/n < b < \rho_+(P)$. By Lemma 3.12, there is a map G which agrees with F on its nonconstant part, such that $\rho_-(G) = a$ and $\rho_+(G) = b$. By Lemma 3.11, G has a cycle Q, with period n and rotation number r/n, which lies on the nonconstant part of G. Thus Q is an F-cycle as well. But $\rho_-(P) < \rho_-(G) \le \rho_-(Q)$. Therefore P and Q are not equivalent. $\square$

An immediate consequence of Lemma 3.13 is

Theorem 3.14. *Primary cycles have trivial rotation intervals.*

Proof of Theorem 3.3. Suppose every cycle of period $|P|$ and rotation number $\rho(P)$ which is forced by P is equivalent to P. Let P force Q, where Q is of period $|P|$. It follows from Lemma 3.6 that $\rho_-(P) \le \rho_-(Q) \le \rho_+(Q) \le \rho_+(P)$. By Lemma 3.13, $\rho_-(P) = \rho_+(P)$. Therefore $\rho(P) = \rho(Q)$, and hence P and Q are equivalent. $\square$

The converse of Theorem 3.14 is false. We construct an example of a cycle with a trivial rotation interval which is not primary.

Example. Let $\varphi: P \to P = \{\cdots < x_0 < x_1 < \cdots\}$ and $\psi: Q \to Q = \{\cdots < y_0 < y_1 < \cdots\}$ be cycles of period 6 and rotation number $\frac{1}{2}$, defined as follows. Let $\alpha(0, 1, 2, 3, 4, 5) = (4, 5, 3, 7, 8, 6)$, i.e., $\alpha(0) = 4$, $\alpha(1) = 5$,

etc., and $\varphi(x_{6i+k}) = x_{6i+\alpha(k)}$. Let $\beta(0,1,2,3,4,5) = (3,5,4,8,7,6)$ and $\psi(y_{6i+k}) = y_{6i+\beta(k)}$. Then P has a trivial rotation interval by Theorem 3.15 below. However, since P forces Q (as can be verified by examining the Markov graph of P), P is not primary.

Definition. A cycle $\varphi: P \to P$ of period n has a *block structure* over a cycle $\psi: Q \to Q = \{\cdots < y_0 < y_1 < \cdots\}$ of period m if n is a multiple of m, say $n = sm$, and P can be written $P = \bigcup_{i \in \mathbb{Z}} P_i$, where $P_i = \{x_{is} < \cdots < x_{(i+1)s-1}\}$, in such a way that $\varphi(P_i) = P_j$ if and only if $\psi(y_i) = y_j$.

Theorem 3.15. *A cycle has a trivial rotation interval if and only if it has a block structure over a twist cycle.*

Proof. Suppose $\varphi: P \to P$ has a trivial rotation interval, and let F be P-monotone. By Lemma 3.8, F has a twist cycle Q such that $\rho(P) = \rho(Q)$, and for all $x \in P$ and $y \in Q$, $F(x) < F(y)$ if and only if $x < y$. Write $Q = \{\cdots < y_0 < y_1 < \cdots\}$ and let $\psi = F|_Q$ and $P_i = P \cap [y_i, y_{i+1})$. Then each block P_i has the same number of points, and $\varphi(P_i) = P_j$ if and only if $\psi(y_i) = \psi(y_j)$. Thus P has a block structure over Q.

Conversely, suppose P has a block structure $P = \bigcup P_i$ over a twist cycle with rotation number r/n (in lowest terms). Let F be P-monotone. Then $F(P_i) = P_{i+r}$ and $F^n(P_i) = P_i + r$. Let $G_i = [\max P_i, \min P_{i+1}]$ be the gap between P_i and P_{i+1}. Then $F(G_i) \supseteq G_{i+r}$ and $F^n(G_i) \supseteq G_i + r$. Hence, by the standard Markov graph argument, there exist $y_i \in G_i$ such that $F(y_i) = y_{i+r}$ and $F^n(y_i) = y_i + r$. Then $Q = \{\cdots < y_0 < y_1 < \cdots\}$ is a twist cycle, and P, Q, and F satisfy the hypotheses of Lemma 3.9. Hence P has a trivial rotation interval. $\square$

4. A CHARACTERIZATION OF PRIMARY CYCLES

Definition. Let $\varphi: P \to P$ be a cycle which has a block structure $P = \bigcup P_i$ over a cycle Q of period n and rotation number r/n, and let X be an interval cycle. P is an *X-extension* of Q if for some k, $0 \le k \le n-1$, φ is monotone on P_i except possibly when $i \equiv k \pmod{n}$, and P_k is a $(\varphi^n - r)$-cycle equivalent to X. As in the interval case, a *2-extension* is one in which X is a two-point cycle, and a *Štefan extension* is one in which X is a Štefan cycle.

If P is an X-extension as above, then for $i \not\equiv k \pmod{n}$, P_i is a $(\varphi^n - r)$-cycle equivalent to either X or its "mirror image"—obtained by conjugating X by an orientation-reversing homeomorphism of the interval, i.e., looking at X upside-down and backward.

Note that if P has a block structure over Q and $|P| = 2|Q|$, then P is a 2-extension of Q.

Example. The cycle $\psi: Q \to Q$ in the example at the end of §3 is a Štefan extension of a twist cycle of period 2 and rotation number $\frac{1}{2}$.

Theorem 4.1. *The following statements about a cycle P are equivalent.*

(1) *P is primary.*

(2) *Every cycle of period $|P|$ and rotation number $\rho(P)$ which is forced by*
P is equivalent to P .

(3) *P can be obtained from a twist cycle by a finite number (possibly zero)*
of 2-extensions and then at most one Štefan extension.

(4) *There is a map in $\mathscr{C}$ whose only cycle of period $|P|$ is P .*

Note that $(4) \Rightarrow (2)$ is obvious and that $(2) \Rightarrow (1)$ is Theorem 3.3. Thus we
need only prove $(3) \Rightarrow (4)$ and $(1) \Rightarrow (3)$.

Proof of Theorem 4.1. $(3) \Rightarrow (4)$. The proof is similar to that of Theorem 1.1,
$(2) \Rightarrow (3)$, with the twist cycle playing the role of the one-point cycle. Let
$\varphi \colon P \to P = \{ \cdots < x_0 < x_1 < \cdots \}$.

If P is a twist cycle, then (4) holds by Theorem 3.2. If P is obtained from a
twist cycle by taking a finite, nonzero number of 2-extensions, renumber P so
that $P = \bigcup P_i$, where $P_i = \{ x_{is} < \cdots < x_{(i+1)s-1} \}$, exhibits the block structure
of P over the twist cycle. In this case, let F agree with φ on P and be
monotone quadratic on $[x_i, x_{i+1}]$ for $i \equiv 0 \pmod{|P|}$ and linear on $[x_i, x_{i+1}]$
for $i \not\equiv 0 \pmod{|P|}$. If P is obtained from a twist cycle by taking a finite
number (possibly zero) of 2-extensions and then taking a Štefan extension, let
F agree with φ on P and be linear on each $[x_i, x_{i+1}]$. In the latter two cases,
it is easy to see, using the Markov graph, that P is the only F -cycle of period
$|P|$. $\square$

It remains only to prove $(1) \Rightarrow (3)$.

Definition. For $A \subseteq \mathbf{R}$ closed and bounded, let $[A] = [\min A, \max A]$ denote
the convex hull of A .

Lemma 4.2. *Let Q be a cycle which has a block structure $Q = \bigcup Q_i$. Suppose*
G is Q -monotone and has a cycle S which has a block structure $S = \bigcup S_i$ such
that $S_i \subseteq [Q_i]$ for each i . If Q is a cycle of a map $F \in \mathscr{C}$, then F has a cycle
R with a block structure $R = \bigcup R_i$ such that $|R| = |S|$ and $R_i \subseteq [Q_i]$ for each
i .

Proof. This follows in the same way as does Lemma 2.2. $\square$

Lemma 4.3 (cf. [ALM, Lemma 3.13]). *Let $\varphi \colon P \to P$ be a primary cycle which*
has a block structure $P = \bigcup P_i$ over a cycle of period n and rotation number
r/n . Then each block P_i is a primary interval cycle of $\varphi^n - r$.

Proof. Suppose not. By renumbering the blocks, we may assume that P_0 is a
$(\varphi^n - r)$ -cycle which is not primary.

By Lemma 2.4, there is a P -monotone map F which has only finitely many
cycles of each period. Since $F^n - r$ maps $[P_0]$ to itself, $F^n - r$ has a cycle
Q_0 of period $|P_0|$ which is properly contained in $[P_0]$. We may choose Q_0
so that $[Q_0]$ contains no other $(F^n - r)$ -cycle of period $|P_0|$. Then $Q =$

$\bigcup_{k \geq 0, s \in \mathbf{Z}}(F^k(Q_0) + s)$ is an F-cycle which has a block structure $Q = \bigcup Q_i$, with each $Q_i \subseteq [P_i]$, over the same cycle as does P. Since $|P| = |Q|$, P and Q are equivalent. The equivalence respects the block structures. Let P_0 correspond to Q_j.

Now apply the argument given above to Q_j. By Lemma 4.2, there is an F-cycle R of period $|P|$, with a block structure $R = \bigcup R_i$ where each R_i is an $(F^n - r)$-cycle of period $|P_0|$, properly contained in $[Q_i]$. This violates the choice of Q_0. □

Lemma 4.4 (cf. [ALM, Lemma 8.2]). *Let P be primary and suppose P has a block structure over Q. If $|P|/|Q|$ is even, then P has a block structure over some 2-extension of Q.*

Proof. Let $\varphi: P \to P = \bigcup P_i$ exhibit the block structure, and suppose Q has period n and rotation number r/n. Each block P_i has $|P|/|Q|$ points.

It suffices to show that $\varphi(\{L_i, R_i\}) = \{L_j, R_j\}$ whenever $\varphi(P_i) = P_j$, where L_i and R_i are the left and right halves of P_i.

Fix i. By Lemma 4.3, P_i is a primary interval cycle of $\varphi^n - r$. By Theorem 1.1, $(1) \Rightarrow (2)$, $\varphi^n - r$ interchanges L_i and R_i, and so for every even integer $k > 0$, $\varphi^{kn} - kr$ maps L_i to L_i and R_i to R_i. Now suppose $\varphi(P_i) = P_j$ and x and y both lie in L_i or both lie in R_i. There is an even integer $k > 0$ such that $y = \varphi^{kn}(x) - kr$. If $\varphi(x) \in L_j$, then so is $\varphi^{kn}(\varphi(x)) - kr = \varphi(y)$. Similarly, if $\varphi(x) \in R_j$, then so is $\varphi(y)$. Thus $\varphi(\{L_i, R_i\}) = \{L_j, R_j\}$. □

Lemma 4.5. *Let f be an interval map which has a cycle X of period sm, where $m \geq 3$ is odd. If X has a block structure $X = X_1 \cup \cdots \cup X_s$ over some cycle of period s, then f has a cycle Y, also of period sm, which has a block structure $Y = Y_1 \cup \cdots \cup Y_s$ over the same cycle as does X, such that each Y_i is a Štefan cycle of f^s of period m, contained in the convex hull of X_i, and except for exactly one i, f maps Y_i monotonically onto $f(Y_i)$.*

Proof. This result is proved in [BC] under the assumption that s is a power of two. An examination of the proof reveals that this assumption is unnecessary. □

Proof of Theorem 4.1. $(1) \Rightarrow (3)$. Let P be primary. By Theorems 3.14 and 3.15, P has a block structure over a twist cycle. Applying Lemma 4.4 the appropriate number of times, we find that P has a block structure $P = \bigcup P_i$ over a cycle Q which is obtained from the twist cycle by taking a finite number (possibly zero) of 2-extensions, and such that $|P|/|Q|$ is odd.

If $|P|/|Q| = 1$, then P is equivalent to Q and we are done. So suppose that $|P|/|Q| = m$, where $m \geq 3$ is odd. Let F be P-monotone and suppose that Q has period s and rotation number t/s. Let $X = P_0 \cup F(P_0) \cup \cdots \cup F^{s-1}(P_0)$, and define $f: [X] \to [X]$ as follows. Let f agree with F on $[P_0] \cup \cdots \cup F^{s-2}([P_0])$, $f(x) = F(x) - t$ on $F^{s-1}([P_0])$, and then extend f to an interval map of $[X]$ to itself which is linear between $F^i([P_0])$ and $F^j([P_0])$ whenever these intervals

are "adjacent." Then f and X satisfy the hypotheses of Lemma 4.5. Let Y be an f-cycle whose existence is guaranteed by Lemma 4.5. Then $R = \bigcup_{k\in\mathbf{Z}}(Y+k)$ is an F-cycle which is a Štefan extension of Q. By Lemma 2.2, P forces R. But P is primary, so P is equivalent to R. Thus P is a Štefan extension of Q. $\square$

An alternate proof of Theorem 4.1, $(1) \Rightarrow (3)$ may be obtained by adapting the lemmas of [ALM, §8] to the circle.

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